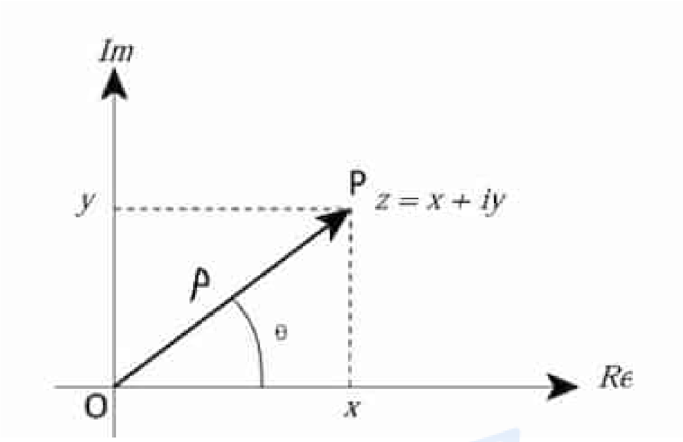
Complex Numbers

Introduction

Representation methods



The methods are:

- Cartesian representation: $z = x + iy$
- Polar representation: $z = pe^{i\theta}$

Here:

- $x = p\cos\theta$ - real part
- $y = p\sin\theta$ - imaginary part
- $p = \sqrt{x^2 + y^2}$ - modulus
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ - arg angle

Euler's Formula

For $x \in \mathbb{R}$:

$$e^{ix} = \cos x + i\sin x$$

> **Proof Hint**
>
> Use Taylor series for $e^x, \cos x, \sin x$.

Euler's Identity

One of the most beautiful equations in mathematics.

$$e^{i\pi} + 1 = 0$$

Roots of Unity

n-th roots of unity (1) are the complex numbers that satisfy the equation, $z^n = 1$. There are n distinct solutions.

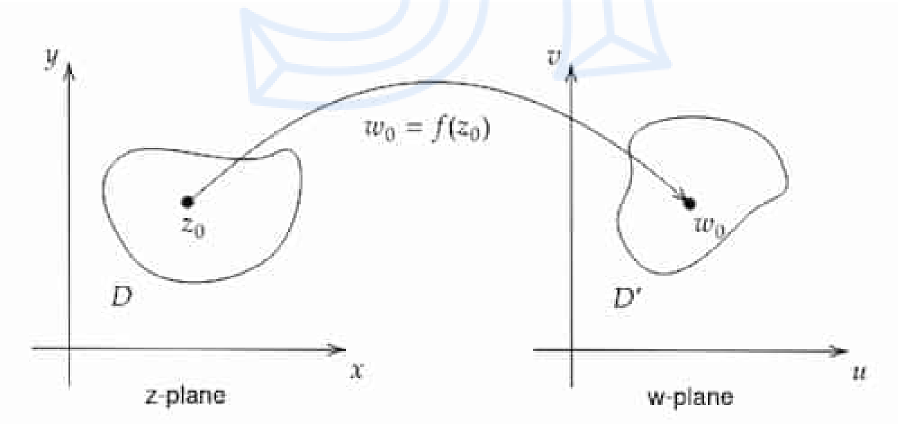
$$z = \exp\left(i\left(\frac{2m\pi}{n}\right)\right) \quad \textbf{where} \quad m \in \mathbb{Z} \cup [0, n)$$

The solution can be written as $1, w, w^2, w^3, \ldots, w^{n-1}$.

1 is called the trivial solution. Other solutions are called as primitive n-th roots.

Complex Functions

Suppose $w = f(z)$ where $z, w \in \mathbb{C}$. Input and output points are marked in 2 separate complex planes.



Here:

- D - domain of f
- D' - codomain of f

Image

Image of f is the set:

$$\{f(z) \mid z \in D\}$$

Cartesian form

$$f(z) = u(x, y) + iv(x, y)$$

Here u, v are real functions.

Limits

$\lim_{z \to z_0} f(z) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall z \; (0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon)$$

Properties

All properties mentioned in [Limits | Real Analysis] are applicable to complex limits. Additional properties are mentioned below:

Suppose $\lim f(z) = L$.

- $\lim \overline{f(z)} = \overline{L}$
- $\lim \mathrm{Re}(f(z)) = \mathrm{Re}(L)$
- $\lim \mathrm{Im}(f(z)) = \mathrm{Im}(L)$

Real and imaginary limits

Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, $z = x + iy$.

Suppose the real part and imaginary part limits to L_1, L_2, which can be written as:

$$\lim_{(x,y) \to (x_0, y_0)} u(x, y) = L_1 \qquad \lim_{(x,y) \to (x_0, y_0)} v(x, y) = L_2$$

Then:

$$\lim_{z \to z_0} f(z) = L_1 + iL_2$$

Difference from real functions

For real functions, when considering the limit at a point, the limit could be be approaching the point either from left or right.

For complex functions, the point can be approached along any path in the complex plane. The distance $|z - z_0|$ decreases to 0.

Notes for questions

- When 2 arbitrary paths are chosen: if the limits on each are different, then the limit DNE.
- When substituting $z = x + imx$: if m doesn't cancel out, then the limit DNE.
- In most limits, subtituting $z = re^{i\theta}$ will simplify the limit a lot.
- In very complex functions, limits can be taken for real and imaginary parts separately.

Important limits

$$\lim_{z \to 0} \frac{z}{\overline{z}} \text{ doesn't exist}$$

The above limit is important as it shows up in many questions. Can be disproved by taking two paths: real, imaginary axes.

$$\lim_{z \to 0} \frac{z\overline{z}}{z + \overline{z}} \text{ doesn't exist}$$

Can be proven usign taking 2 paths: real axis, $t + \sqrt{t}i$.

Continuity

$f(z)$ is continuous at z_0 **iff**:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

$$\iff \forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \, \left(|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon \right)$$

Conditions

For f to be continuous on z_0, all these conditions are required.

- f is defined at z_0
- $\lim_{z \to z_0} f(z)$ exists
- $\lim_{z \to z_0} f(z) = f(z_0)$

Properties

If f, g are continuous at z_0, these functions are also continuous at z_0:

- $\operatorname{Re}(f)$
- $\operatorname{Im}(f)$
- $|f|$
- $f \pm g$
- fg
- $\frac{f}{g}$ where $g \neq 0$
- $f(g(z))$

Differentiability

A complex function f is differentiable at z_0 **iff**:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L = f'(z_0)$$

$f'(z_0)$ is called the derivative of f at z_0. Properties of Differentiability | Real Analysis can be applied to complex functions.

Singular point

A point z_0 where $f(z)$ is not differentiable.

Neighbourhood

Suppose $z_0 \in \mathbb{C}$. A neighborhood of z_0 is the region contained in the circle $|z - z_0| = r > 0$.

Analytic Functions

A function f is said to be analytic at z_0 **iff** it is differentiable throughout a neighbourhood of z_0. Aka. regular functions, holomorphic functions and monogenic functions.

Examples

- Polynomial functions of z (analytic everywhere)
- Functions with a converging Taylor series for all z (analytic everywhere)
  - $\sin z$
  - $\cos z$
  - e^z

Non-examples

Function	Note
$\|z\|^2$	Differentiable only at $z = 0$.
$\overline{z}$	Nowhere differentiable. Derivative taken on the real and imaginary axes are different.
$\text{Re}(z)$	Similar to above.
$z + \overline{z}$	Similar to above.
$\text{Im}(z)$	Similar to above.
$z - \overline{z}$	Similar to above.

Analytic implies differentiable

$$f \text{ is analytic at } z_0 \implies f \text{ is differentiable at } z_0$$

Cauchy-Riemann Equations

Suppose f is a complex-valued function of a complex variable. **If** the derivatives are the same for the 2 paths —real and imaginary axes— **then** f is analytic.

Suppose $f(z) = u(x, y) + iv(x, y)$ for the theorems below.

The equations

The set of equations mentioned below are the Cauchy Riemann Equations, where u, v are functions of x, y.

Cartesian form

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial v}{\partial y} = v_y \quad \wedge \quad \frac{\partial u}{\partial y} = u_y = -\frac{\partial v}{\partial x} = -v_x$$

Polar form

Here the partial derivatives are about r, θ.

$$u_r = \frac{1}{r} v_\theta \quad \wedge \quad v_r = -\frac{1}{r} u_\theta$$

Complex form

$$f_x = -i f_y$$

Theorem 1

If f is differentiable at z_0, **then**

- All partial derivatives u_x, u_y, v_x, v_y exist **and**
- They satisfy the Cauchy Riemann equations

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$$

> ⓘ **Note**
>
> Contrapositive is useful when proving f is **not** differentiable at z_0.

Theorem 2

If:

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- All partial derivatives u_x, u_y, v_x, v_y exist **and**
- They satisfy Cauchy-Riemann equations **and**
- They are continuous at z_0

Then:

- f is differentiable at z_0 **and**

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

Theorem 3

If f is analytic at z_0, **then** its first-order partial derivatives are continuous in a neighbourhood of z_0.

Entire Functions

A complex function that is differentiable everywhere. Which implies that they are analytic everywhere.

Examples:

- polynomial functions
- $\sin z, \cos z, e^z$

Non-examples:

- Rational functions

From Sahithyan's S1