

# Differential Equations

## Introduction

Equations which are composed of an unknown function and its derivatives.

## Types

### Ordinary Differential Equations

When a differential equation involves one independent variable, and one or more dependent variables.

An example:

$$\frac{dy}{dx} = \cos(x)$$

### Partial Differential Equations

When a differential equation involves more than one independent variables, and more than one dependent variables.

$$\frac{\partial y}{\partial x} = y_x = \cos(x)$$

## Linear

A linear differential equation is a differential equation that is defined by a linear polynomial in the unknown function (dependent variable) and its derivatives, that is an equation of the form:

$$P_0(x)y + P_1(x)y' + \dots + P_n(x)y^{(n)} + Q(x) = 0$$

Here:

- $P_0, P_1, \dots, P_n, Q$  are all differentiable functions of  $x$ , doesn't depend on  $y$
- $y(x)$  is the unknown function
- $y^{(n)}$  denotes the  $n$ th derivative of  $y$

## Nonlinear

Nonlinear differential equations are any equations that cannot be written in the above form. In particular, these include all equations that include:

- $y$  and/or its derivatives raised to any power other than 1
- nonlinear functions of  $y$  or any of its derivative
- any product or function of these

## Properties of Differential Equations

### Order

Highest order derivative.

### Degree

Power of highest order derivative.

### Initial Value Problem (IVP)

A differential equation along with appropriate number of initial conditions.

Initial condition(s) is/are required to determine which solution (out of the infinite number of solutions) is the suitable one for the given problem.

### Picard's Existence and Uniqueness Theorem

Consider the below IVP.

$$\frac{dy}{dx} = f(x, y); y(x_0) = y_0$$

Suppose:  $D$  is an open neighbourhood in  $\mathbb{R}^2$  containing the point  $(x_0, y_0)$ .

**If**  $f$  and  $\frac{\partial f}{\partial y}$  are continuous functions in  $D$ , **then** the IVP has a unique solution in some closed interval containing  $x_0$ .

# Solving First Order Ordinary Differential Equations

## Separable equation

When  $x$  and  $y$  functions can be separated into separate one-variable functions (as shown below).

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

## Homogenous equation

$$\frac{dy}{dx} = f(x, y)$$

A function  $f(x, y)$  is homogenous when  $f(x, y) = f(\lambda x, \lambda y)$ .

To solve:

- Use  $y = vx$  substitution, where  $v$  is a function of  $x$  and  $y$
- Differentiate both sides:  $dy = v + vdx$
- Apply the substitution to make it separable

## Reduction to homogenous type

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$$

This type of equation can be reduced to homogenous form.

If  $a : b = A : B$ , use the substitution:  $u = ax + by$ .

In other cases:

- Find  $h$  and  $k$  such that  $ah + bk + c = 0$  and  $Ah + Bk + C = 0$
- Use substitutions:
  - $X = x + h$
  - $Y = y + k$

The reduced equation would be:

$$\frac{dY}{dX} = \frac{aX + bY}{AX + BY}$$

## Linear equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

The above form is called **the standard form**.

The equation would be separable if  $Q(x) = 0$ .

Otherwise:

- Identify  $P(x)$  from the standard form
- Calculate **integrating factor**:  $I = \exp \int P(x) dx$ .
- Multiply both sides by  $I$
- **LHS** becomes  $\frac{d}{dx}(yI)$
- Integrate both sides to solve for  $y$

## Bernoulli's equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n ; n \in \mathbb{R}$$

When  $n = 0$  or  $n = 1$ , the equation would be linear.

Otherwise, it can be converted to linear using  $v = y^{1-n}$  as substitution.

## None of the above

The equation must be converted to one of the above by using a suitable substitution.

# Higher Order Ordinary Differential Equations

## Linear Differential Equations

$$\frac{d^n y}{dx^n} + p_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + p_n(x) y = q(x)$$

Based on  $q(x)$ , the above equation is categorized into 2 types:

- **Homogenous** if  $q(x) = 0$
- **Non-homogenous** if  $q(x) \neq 0$

① For 1st semester

Only linear, ordinary differential equations with constant coefficients are required.

They can be written as:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = q(x)$$

## Solution

The general solution of the equation is  $y = y_p + y_c$ .

Here

- $y_p$  - **particular solution**
- $y_c$  - **complementary solution**

## Particular solution

Doesn't exist for homogenous equations. For non-homogenous equations check [steps section of 2nd order ODE](#).

## Complementary solution

Solutions assuming  $LHS = 0$  (as in a homogenous equation).

$$y_c = \sum_{i=1}^n c_i y_i$$

Here

- $c_i$  - constant coefficients
- $y_i$  - a linearly-independent solution

## Linearly dependent & independent

$n$ -th order linear differential equations have  $n$  linearly independent solutions.

Two solutions of a differential equation  $u, v$  are said to be **linearly dependent**, if there exists constants  $c_1, c_2$  ( $\neq 0$ ) such that  $c_1 u(x) + c_2 v(x) = 0$ .

Otherwise, the solutions are said to be **linearly independent**, which means:

$$\sum_{i=1}^n c_i y_i = 0 \implies \forall c_i = 0$$

## Linear differential operators with constant coefficients

⚠ WTF?

I don't understand anything in this section.

## Differential operator

Defined as:

$$D^i = \frac{d^i}{dx^i}; n \in \mathbb{Z}^+$$

The above equation can be written using the differential operator:

$$D^n y + a_1 D^{n-1} y + \dots + a_n y = q(x)$$

Here if  $y$  is factored out (**how tf?**):

$$(D^n + a_1 D^{n-1} + \dots + a_n) y = P(D) y = q(x)$$

where  $P(D) = (D^n + a_1 D^{n-1} + \dots + a_n)$ .

$P(D)$  is called a polynomial differential operator with constant coefficients.

## Solving Second Order Ordinary Differential Equations

### Homogenous

$$\frac{d^2 y}{dx^2} + a \frac{dy}{dx} + by = 0; \quad a, b \text{ are constants}$$

Consider the function  $y = e^{mx}$ . Here  $m$  is a constant to be found.

By applying the function to the above equation:

$$m^2 + am + b = 0$$

The above equation is called the **Auxiliary equation** or **Characteristic equation**.

#### Case 1: Distinct real roots

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

#### Case 2: Equal real roots

$$y = (Ax + B)e^{mx}$$

#### Case 3: Complex conjugate roots

$$y = Ae^{(p+iq)x} + Be^{(p-iq)x} = e^{px} (C \cos(qx) + D \sin(qx))$$

## Non-homogenous

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = q(x); \quad a, b \text{ are constants}$$

### Method of undetermined coefficients

$y_p$  is guessed with undetermined coefficients. And substituted in the given equation to find the coefficients. The guess depends on the nature of  $q(x)$ .

If  $q(x)$  is:

- a constant,  $y_p$  is a constant
- $kx$ ,  $y_p = ax + b$
- $kx^2$ ,  $y_p = ax^2 + bx + c$
- $k \sin x$  or  $k \cos x$ ,  $y_p = a \sin x + b \cos x$
- $e^{kx}$ ,  $y_p = ce^{kx}$  (Only works if  $k$  is **not** a root of auxiliary equation)
- A product of  $e^{kx}$  and some  $f(x)$ , guess  $y_p$  for  $f(x)$  individually, and then multiply by  $e^{kx}$  (without coefficients)
- A product of polynomials and trig functions, guess  $y_p$  for the polynomial, and multiply that by the appropriate cosine. Then add on a new guess for the polynomial with different coefficients and multiply that by the appropriate sine.
- A sum of functions, can be guessed individually and be summed up

### Steps

- Solve for  $y_c$
- Based on the form of  $q(x)$ , make an initial guess for  $y_p$ .
- Check if any term in the guess for  $y_p$  is a solution to the complementary equation.
- If so, multiply the guess by  $x$ . Repeat this step until there are no terms in  $y_p$  that solve the complementary equation.
- Substitute  $y_p$  into the differential equation and equate like terms to find values for the unknown coefficients in  $y_p$ .
- If coefficients were unable to be found (they cancelled out or something like that), multiply the guess by  $x$  and start again.
- $y = y_p + y_c$



# Wronskian

Consider the equation, where  $P, Q$  are functions of  $x$  alone, and which has 2 fundamental solutions  $u(x), v(x)$ :

$$y'' + Py' + Qy = 0$$

The Wronskian  $w(x)$  of two solutions  $u(x), v(x)$  of differential equation, is defined to be:

$$w(x) = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix}$$

## Theorem 1

Suppose  $u$  and  $v$  are two solutions of a 2nd order differential equation, in the form mentioned above.

$W(u, v)$  is **always zero or never zero** (in the intended range of solutions).

### ⓘ Note

If the Wronskian is non-zero at a point and zero on another point,  $u, v$  cannot be solutions to the same differential equation.

## Proof

Consider the equation, where  $P, Q$  are functions of  $x$  alone.

$$y'' + Py' + Qy = 0$$

Let  $u(x), v(x)$  be 2 fundamental solutions of the equation:

$$u'' + Pu' + Qu = 0 \quad \wedge \quad v'' + Pv' + Qv = 0$$

$$w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$$

$$w' = uv'' - vu'' = -P[uv' - vu'] = -Pw$$

By solving the above relation:

$$w = c \cdot \exp\left(-\int P dx\right)$$

Suppose there exists  $x_0$  such that  $w(x_0) = 0$ . That implies  $c = 0$ . That implies  $w$  is always 0.

## Abel's formula

The conclusion in the above proof is known as the Abel's formula.

$$W(x) = c \exp\left(-\int P dx\right)$$

## Theorem 2

$$\{f_1, f_2, \dots, f_n\} \text{ are linearly dependent} \iff \forall x; W(f_1, f_2, \dots, f_n)(x) = 0$$

### Note

This theorem is valid only under some conditions, which are out of the scope of the course and not covered.

## Variation of parameters

Consider the equation, where  $P, Q$  are functions of  $x$  alone, and which has 2 fundamental solutions  $y_1, y_2$ :

$$y'' + Py' + Qy = f(x)$$

The general solution of the equation is:

$$y_g = c_1 y_1 + c_2 y_2$$

Now replacing  $c_1, c_2$  with  $u(x), v(x)$  will give  $y_p = uy_1 + vy_2$ , which can be found using the method of variation of parameters.

$$u = - \int \frac{y_2 f}{W(x)} dx \wedge v = \int \frac{y_1 f}{W(x)} dx$$

## Proof

$$y_p = uy_1 + vy_2$$

$$y'_p = u'y_1 + uy'_1 + v'y_2 + vy'_2$$

Set  $u'y_1 + v'y_2 = 0$  (1) to simplify further equations. That implies  $y'_p = uy'_1 + vy'_2$ .

$$y''_p = uy''_1 + u'y'_1 + vy''_2 + v'y'_2$$

Substituting  $y''_p, y'_p, y_p$  to the differential equation:

$$y''_p + Py'_p + Qy_p = u'y'_1 + v'y'_2$$

This implies  $u'y'_1 + v'y'_2 = f(x)$  (2).

From equations (1) and (2), where  $W(x)$  is the wronskian of  $y_1, y_2$ :

$$u' = -\frac{y_2 f}{W(x)} \wedge v' = \frac{y_1 f}{W(x)}$$

$$u = - \int \frac{y_2 f}{W(x)} dx \wedge v = \int \frac{y_1 f}{W(x)} dx$$

$y_p$  can be found now using  $u, v, y_1, y_2$