Vectors

Introduction

Revise Vectors unit from G.C.E (A/L) Combined Mathematics.

Direction Cosines

Suppose $\vec{p} = a\underline{i} + b\underline{j} + c\underline{k}$. Direction cosines of p are $\cos\alpha, \cos\beta, \cos\gamma$ where α, β, γ are the angles p makes with x, y, z axes.

Unit vector in the direction of $\vec{p} = \underline{i}\cos\alpha + \underline{j}\cos\beta + \underline{k}\cos\gamma$. Because of this:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Direction Ratio

Ratio of the direction cosines is called as direction ratio.

$$\cos\alpha \ : \ \cos\beta \ : \ \cos\gamma$$

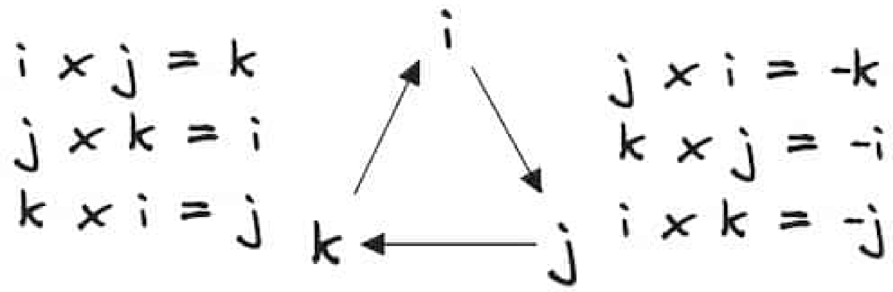
Cross Product

$$a \times b = |a||b|sin(\theta)n = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

n is the **unit normal vector** to a and b. Direction is based on the right hand rule.

$$a \times b = 0 \implies |a| = 0 \lor |b| = 0 \lor a \parallel b$$

Cross products between i, j, k are circular.



Properties

- $a \times a = 0$
- $(a \times b) = -(b \times a)$
- $a \times (b + c) = (a \times b) + (a \times c)$
- Area of a parallelogram $ABCD = |\vec{AB} \times \vec{AD}|$

Scalar Triple Product

$$[a, b, c] = a \cdot (b \times c) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Properties

- $[a, b, c] = a \cdot (b \times c) = (a \times b) \cdot c$
- $[a, b, c] = [b, c, a] = [c, a, b] = -[a, c, b]$
- Swapping any 2 vectors will negate the product.
- $[a, b, c] = 0$ **iff** a , b , c are coplanar.
- Volume of a parallelepiped with a , b , c as adjacent edges = $[a, b, c]$
- Volume of a tetrahedron with a , b , c as adjacent edges = $\frac{1}{6}[a, b, c]$

Vector Triple Product

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

Resulting vector lies in the plane that contains b and c.

Section Formula

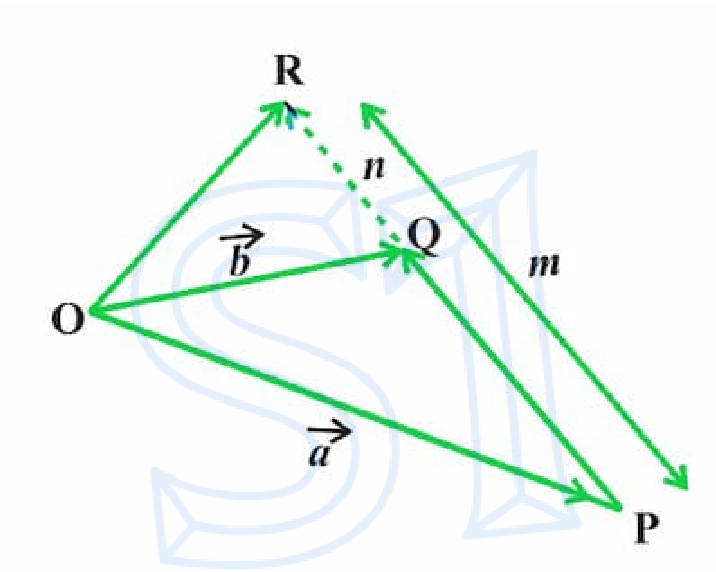
Suppose R divides the line segment PQ in the ratio $m : n$ (both are positive and $m \geq n$), the division can either be internal or external.

Internally

Internal section

$$\overrightarrow{\mathrm{OR}} = \frac{m\overrightarrow{\mathrm{OQ}} + n\overrightarrow{\mathrm{OP}}}{m+n}$$

Externally



$$\overrightarrow{\mathrm{OR}} = \frac{m\overrightarrow{\mathrm{OQ}} - n\overrightarrow{\mathrm{OP}}}{m-n}$$

Straight Lines

Passes through a point & parallel to a vector

Equation for a line that:

- passes through $\underline{r_0} = \langle x_0, y_0, z_0 \rangle$
- is parallel to $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$

Parametric equation

$$\underline{r} = \underline{r_0} + t\underline{v};\ t \in \mathbb{R}$$

Symmetric equation

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Passes through 2 points

Equation of a line passes through $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$. $\underline{r_A}$ and $\underline{r_B}$ are the position vectors of A and B.

Parametric equation

$$\underline{r} = (1 - t)\underline{r_A} + t\underline{r_B};\ t \in \mathbb{R}$$

Symmetric equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Intersection

To show that two straight lines intersect in 3D space, existence of a point which satisfies both lines must be proven.

It is **not** enough to show that the cross product of their parallel vectors is non-zero.

Normal to 2 lines

Let α, β be two lines.

$$\alpha : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1};\quad \beta : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

Here $v_1 = \langle a_1, b_1, c_1 \rangle$, $v_2 = \langle a_2, b_2, c_2 \rangle$ are 2 vectors parallel to α, β respectively.

Normal to both lines: $v_1 \times v_2$.

Unit normal

$$\frac{v_1 \times v_2}{|v_1 \times v_2|}$$

Angle between 2 straight lines

Using the α, β lines mentioned above:

$$\cos\theta = \frac{v_1 \cdot v_1}{|v_1| \cdot |v_2|} = \frac{(a_1\underline{i} + b_1\underline{j} + c_1\underline{k}) \cdot (a_2\underline{i} + b_2\underline{j} + c_2\underline{k})}{|a_1\underline{i} + b_1\underline{j} + c_1\underline{k}| \cdot |a_2\underline{i} + b_2\underline{j} + c_2\underline{k}|}$$

Here v_1, v_2 are 2 vectors parallel to α, β respectively.

Shortest distance from a point

The distance can be calculated using Pythogoras' theorem.

$$d^2 = \left|\underline{r} - \underline{p}\right|^2 - \left[\frac{\underline{n} \cdot (\underline{r} - \underline{p})}{|\underline{n}|}\right]^2$$

Here:

- P is the arbitrary point
- $\underline{p}$ is the position vector of P
- $\underline{r}$ is the position vector of a point on the line
- $\underline{n}$ is parallel to the line

Planes

Equation of planes can expressed in either vector or cartesian form. Vector equation is the one containing only vectors. Cartesian equation is in the form: $Ax + By + Cz = D$.

Contains a point and parallel to 2 vectors

Suppose a plane:

- is parallel to both $\underline{a}$ and $\underline{b}$ where $a \times b \neq 0$
- contains $\underline{r_0} = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$

Equation for the plane is:

$$\underline{r} = \underline{r_0} + s\underline{a} + t\underline{b}\ ;\ \ s, t \in \mathbb{R}$$

Contains a point and normal is given

Suppose a plane:

- contains $\underline{r_0} = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$
- has a normal $\underline{n}$

Equation for the plane is:

$$(\underline{r} - \underline{r_0}) \cdot \underline{n} = 0$$

Contains 3 points

Suppose a plane contains r_0, r_1, r_2 ($\underline{r_0}, \underline{r_1}, \underline{r_2}$ are the position vectors of respectively).

$$(\underline{r} - \underline{r_1}) \ \cdot \ \left[(\underline{r_1} - \underline{r_0}) \times (\underline{r_1} - \underline{r_2})\right] = 0$$

Normal to a plane

Suppose $ax + by + cz = d$ is a plane. $\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$ is a normal to the plane.

Angle between 2 planes

Consider the two planes:

- $A : a_1x + a_2y + a_3z = d$
- $B : b_1x + b_2y + b_3z = d'$

The angle between the planes ϕ is given by:

$$\cos(\phi) = \frac{\underline{n_A} \cdot \underline{n_B}}{|\underline{n_A}| \cdot |\underline{n_B}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}}$$

Here $\underline{n}_A, \underline{n}_B$ are normal to the planes A, B.

Shortest distance from a point

Consider the plane $ax + by + cz = d$.

$$\text{distance} = \frac{\left|(\underline{r}_1 - \underline{r}_0) \cdot \underline{n}\right|}{|\underline{n}|}$$

- $\underline{n}$ is a normal to the plane
- $\underline{r}_0$ is the position vector of any known point on the plane
- $\overline{r}_1$ is the position vector to the arbitrary point

Intersection

In 3D, to prove 2 planes intersect, it has to be proven that there is a point satisfiying both of the planes.

Of 2 planes

Can either be a:

- Plane - when the planes coincicde
- Line - otherwise

Equation of the line of intersection can be found by:

- Solving y, z with respect to x
- Subject x and symmetric form can be found

Of 3 planes

Can either be a:

- Plane - when the planes coincide
- Line - when the lines of intersection between the planes pairwise coincide
- Point - otherwise

First pairwise intersection of the planes must be found. And then intersection of those 2 can be found.

Skew Lines

Two non-parallel lines in 3D-space that do not intersect.

Normal to 2 skew lines

Similar to Normal to 2 lines — Straight Lines.

Distance between 2 skew lines

$$\text{distance} = |\overrightarrow{AB} \cdot \underline{n}|$$

Here

- $\underline{n}$ is the unit normal to both l_1, l_2
- A and B are points lying on each line