Summary | Vectors

Introduction

Revise Vectors unit from G.C.E (A/L) Combined Mathematics.

Cross Product

$$a imes b = |a||b|sin(heta)n = \detegin{pmatrix} i & j & k \ a_x & a_y & a_z \ b_x & b_y & b_z \end{pmatrix}$$

n is the **unit normal vector** to *a* and *b*. Direction is based on the right hand rule.

$$a imes b = 0 \implies |a| = 0 \lor |b| = 0 \lor a \parallel b$$

Cross products between i, j, k are circular.



(i) Note

Area of a parallelogram $ABCD = |ec{AB} imes ec{AD}|$

Scalar Triple Product

$$[a,b,c]=a\cdot(b imes c)=\detegin{pmatrix}a_x&a_y&a_z\b_x&b_y&b_z\c_x&c_y&c_z\end{pmatrix}$$
 $[a,b,c]=a\cdot(b imes c)=(a imes b)\cdot c$ $[a,b,c]=[b,c,a]=[c,a,b]=-[a,c,b]$

[a,b,c] = 0 iff a, b, c are coplanar. Swapping any 2 vectors will negate the product.

(i) Note

Volume of a parallelepiped with a, b, c as adjacent edges = [a, b, c]

Volume of a tetrahedron with a, b, c as adjacent edges = $\frac{1}{6}[a,b,c]$

Vector Triple Product

$$a imes (b imes c) = (a \cdot c)b - (a \cdot b)c$$

Resulting vector lies in the plane that contains $m{b}$ and $m{c}$

Vector Equation of Straight Lines

Passes through a point & parallel to a vector

Equation for a line that:

- passes through $\, \underline{r_0} = \langle x_0, y_0, z_0
 angle \,$
- is parallel to $\, \underline{v} = a \underline{i} + b j + c \underline{k} \,$

Parametric equation

$$\underline{r}=r_{0}+t\overline{v};\;t\in\mathbb{R}$$

Symmetric equation

$$rac{x-x_0}{a}=rac{y-y_0}{b}=rac{z-z_0}{c}$$

Passes through 2 points

Equation of a line passes through $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$. $\underline{r_A}$ and $\underline{r_B}$ are the position vectors of A and B.

Parametric equation

$$\underline{r}=(1-t)r_A+tr_B;\;t\in\mathbb{R}$$

Symmetric equation

$$rac{x-x_1}{x_2-x_1} = rac{y-y_1}{y_2-y_1} = rac{z-z_1}{z_2-z_1}$$

(i) Note

To show that two straight lines intersect in 3D space, it is **not** enough to show that the cross product of their parallel vectors is non-zero.

Also: Existence of a point which satisfies both lines must be proven.

Normal to 2 lines

Let α, β be two lines.

$$lpha: rac{x-x_1}{a_1} = rac{y-y_1}{b_1} = rac{z-z_1}{c_1}; \;\; eta: rac{x-x_2}{a_2} = rac{y-y_2}{b_2} = rac{z-z_2}{c_2}$$

Here $v_1=\langle a_1,b_1,c_1
angle$, $v_2=\langle a_2,b_2,c_2
angle$ are 2 vectors parallel to lpha,eta respectively. Normal to both lines: $v_1 imes v_2$. Unit normal to both lines can be found by:

$$\frac{v_1 \times v_2}{|v_1 \times v_2|}$$

Angle between 2 straight lines

Using the α, β lines mentioned above:

$$\cos\theta = \frac{v_1 \cdot v_1}{|v_1| \cdot |v_2|} = \frac{(a_1\underline{i} + b_1\underline{j} + c_1\underline{k}) \cdot (a_2\underline{i} + b_2\underline{j} + c_2\underline{k})}{|a_1\underline{i} + b_1\underline{j} + c_1\underline{k}| \cdot |a_2\underline{i} + b_2\underline{j} + c_2\underline{k}|}$$

Here v_1, v_2 are 2 vectors parallel to lpha, eta respectively.

Shortest distance to a point

Suppose x_1 and x_2 lie on a line. Shortest distance to the point P is:

$$d^2 = rac{\left| (\underline{x_2} - \overrightarrow{OP}) imes (\underline{x_1} - \overrightarrow{OP})
ight|^2}{\left| \underline{x_2} - \underline{x_1}
ight|^2}$$

Vector Equation of Planes

Contains a point and parallel to 2 vectors

Suppose a plane:

- is parallel to both $\, \underline{a} \,$ and $\, \underline{b} \,$
- contains $\underline{r_0} = x_0 \underline{i} + y_0 \underline{j} + z_0 \underline{k}$

Equation for the plane is:

$$\underline{r}=r_{0}+s\underline{a}+t\underline{b}\ ;\ s,t\in\mathbb{R}$$

Contains a point and normal is given

Suppose a plane:

- contains $\underline{r_0} = x_0 \underline{i} + y_0 \underline{j} + z_0 \underline{k}$
- has a normal $\, \underline{n} \,$

Equation for the plane is:

$$(\underline{r}-r_0)\cdot \underline{n}=0$$

Contains 3 points

Suppose a plane contains r_0, r_1, r_2 ($\underline{r_0}, \underline{r_1}, \underline{r_2}$ are the position vectors of respectively).

$$(\underline{r} - \underline{r_1}) \cdot \left[(\underline{r_1} - \underline{r_0}) \times (\underline{r_1} - \underline{r_2}) \right] = 0$$

Normal to a plane

Suppose ax + by + cz = d is a plane. $\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$ is a normal to the plane.

Angle between 2 planes

Consider the two planes:

- $A: a_1x + a_2y + a_3z = d$
- $B: b_1x + b_2y + b_3z = d'$

The angle between the planes ϕ is given by:

$$\cos(\phi) = rac{n_A \cdot n_B}{|n_A| \cdot |n_B|} = rac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}}$$

Here n_A, n_B are normal to the planes A, B.

Shortest distance to a point

Considering a plane ax + by + cz = d.

$$ext{distance} = rac{|(\underline{r_1} - \underline{r_0}) \cdot \underline{n}|}{|\underline{n}|}$$

- \underline{n} is a normal to the plane
- r_0 is the position vector of a point on the plane
- $\underline{r_1}$ is the position vector to the arbitrary point

Skew Lines

Two non-parallel lines in a 3-space that do not intersect.

Normal to 2 skew lines

Let l_1, l_2 be 2 skew lines.

$$l_1: rac{x-x_0}{a_0} = rac{y-y_0}{b_0} = rac{z-z_0}{c_0} \; ; \; \; l_2: rac{x-x_1}{a_1} = rac{y-y_1}{b_1} = rac{z-z_1}{c_1}$$

The unit normal to both lines n is:

$$\underline{n} = rac{\langle a_0, b_0, c_0
angle imes \langle a_1, b_1, c_1
angle}{|\langle a_0, b_0, c_0
angle imes \langle a_1, b_1, c_1
angle|}$$

Distance between 2 skew lines

$$ext{distance} = |\overrightarrow{AB} \cdot \underline{n}|$$

Here

- \underline{n} is the normal to both l_1, l_2 A and B are points lying on each line

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