Summary | Real Analysis

Introduction— |

| $\land$ | and | | $\lor$ | or | | $\to$ | then | | $\implies$ | implies | | $\Leftarrow$ | implied by | | $\iff$ | if and only if | | $\forall$ | for all | | $\exists$ | there exists | | $\sim$ | not |

Let's take $a \to b$.

1. Contrapositive or transposition: $\sim b \to \sim a$. This is equivalent to the original.
2. Inverse: $\sim a \to \sim b$. Does not depend on the original.
3. Converse: $b \to a$. Does not depend on the original.

$$a \to b \equiv \sim a \lor b \equiv \sim b \to \sim a$$

Examples

- $\sim \forall x P(x) \equiv \exists x \sim P(x)$
- $\sim \exists x P(x) \equiv \forall x \sim P(x)$
- $\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$
- $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$
- $\exists x \forall y P(x, y) \implies \forall y \exists x P(x, y)$
- $(A \to C) \land (B \to C) \equiv (A \lor B) \to C$

Methods of proofs

1. Just proof what should be proven
2. Prove the contrapositive.
3. Proof by contradiction

Proof by contradiction

Let's say we have to prove: $a \implies b$. We will prove $a \land \sim b$ to be false. Then by proof by contradiction, we can prove $a \implies b$.

Proof of proof by contradiction

$$a \wedge \sim b = F$$

$$\sim (a \wedge \sim b) = \sim F$$

$$\sim a \vee b = T$$

$$a \to b = T$$

$$a \implies b$$

Set theory

Zermelo-Fraenkel set theory with axiom of Choice(ZFC):9 axioms all together is being used here.

Definitions

- $x \in A^{\mathrm{c}} \iff x \notin A$
- $x \in A \cup B \iff x \in A \vee x \in B$
- $x \in A \cap B \iff x \in A \wedge x \in B$
- $A \subset B = \forall x (x \in A \implies x \in B)$
- $A - B = A \cap B^{\mathrm{c}}$
- $A = B \iff ((\forall z \in A \implies z \in B) \wedge (\forall z \in B \implies z \in A))$

Required proofs

- $(A \cap B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B^{\mathrm{c}}$
- $(A \cup B)^{\mathrm{c}} = A^{\mathrm{c}} \cap B^{\mathrm{c}}$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \subset A \cup B$
- $A \cap B \subset A$

Set of Numbers

Sets of numbers

- Positive integers: $\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$.
- Natural integers: $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$.
- Negative integers: $\mathbb{Z}^- = \{-1, -2, -3, -4, \ldots\}$.
- Integers: $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$.
- Rational numbers: $\mathbb{Q} = \left\{ \frac{p}{q} \middle| q \neq 0 \land p, q \in \mathbb{Z} \right\}$.
- Irrational numbers: limits of sequences of rational numbers (which are not rational numbers)
- Real numbers: $\mathbb{R} = \mathbb{Q}^c \cup \mathbb{Q}$.

Complex numbers are not part of the study here.

Continued Fraction Expansion

The process

- Separate the integer part
- Find the inverse of the remaining part. Result will be greated than 1.
- Repeat the process for the remaining part.

Finite expansion

Take $\frac{420}{69}$ for example.

$$\frac{420}{69} = 6 + \frac{6}{69}$$

$$\frac{420}{69} = 6 + \frac{1}{\frac{69}{6}}$$

$$\frac{420}{69} = 6 + \frac{1}{11 + \frac{3}{6}}$$

$$\frac{420}{69} = 6 + \cfrac{1}{11 + \cfrac{1}{2}}$$

As $\frac{420}{69}$ is finite, its continued fraction expansion is also finite. And it can be written as $\frac{420}{69} = [6; 11, 2]$.

Infinite expansion

For irrational numbers, the expansion will be infinite.

For example π:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}$$

Conintued fraction expansion of π is $[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \ldots]$.

Field Axioms

Field Axioms of $\mathbb{R}$

$\mathbb{R} \neq \emptyset$ with two binary operations $+$ and $\cdot$ satisfying the following properties

1. Closed under addition: $\forall a, b \in \mathbb{R}; a + b \in \mathbb{R}$
2. Commutative: $\forall a, b \in \mathbb{R}; a + b = b + a$
3. Associative: $\forall a, b, c \in \mathbb{R}; (a + b) + c = a + (b + c)$
4. Additive identity: $\exists 0 \in \mathbb{R}\, \forall a \in \mathbb{R}; a + 0 = 0 + a = a$
5. Additive inverse: $\forall a \in \mathbb{R}\, \exists (-a); a + (-a) = (-a) + a = 0$
6. Closed under multiplication: $\forall a, b \in \mathbb{R}; a \cdot b \in \mathbb{R}$
7. Commutative: $\forall a, b \in \mathbb{R}; a \cdot b = b \cdot a$
8. Associative: $\forall a, b, c \in \mathbb{R}; (a \cdot b) \cdot c = a \cdot (b \cdot c)$
9. Multiplicative identity: $\exists 1 \in \mathbb{R}\, \forall a \in \mathbb{R}; a \cdot 1 = 1 \cdot a = a$
10. Multiplicative inverse: $\forall a \in \mathbb{R} - \{0\}\, \exists a^{-}; a \cdot a^{-} = a^{-} \cdot a = 1$

11. Multiplication is distributive over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$

> ⓘ **Field**
>
> Any set satisfying the above axioms with two binary operations (commonly $+$ and $\cdot$) is called a **field**. Written as $(\mathbb{R}, +, \cdot)$ is a Field. But $(\mathbb{R}, \cdot, +)$ is not a field.

Required proofs

The below mentioned propositions can and should be proven using the above-mentioned axioms. $a, b, c \in \mathbb{R}$.

- $a \cdot 0 = 0$
  Hint: Start with $a(1 + 0)$
- $1 \neq 0$
- Additive identity (0) is unique
- Multiplicative identity (1) is unique
- Additive inverse ($-a$) is unique for a given a
- Multiplicative inverse (a^{-1}) is unique for a given a
- $a + b = 0 \implies b = -a$
- $a + c = b + c \implies a = b$
- $-(a + b) = (-a) + (-b)$
- $-(-a) = a$
- $ac = bc \implies a = b$
- $ab = 0 \implies a = 0 \vee b = 0$
- $-(ab) = (-a)b = a(-b)$
- $(-a)(-b) = ab$
- $a \neq 0 \implies (a^{-1})^{-1} = a$
- $a, b \neq 0 \implies ab^{-1} = a^{-1}b^{-1}$

Field or Not?

	Is field?	Reason (if not)
$(\mathbb{R}, +, \cdot)$	True	
$(\mathbb{R}, \cdot, +)$	False	Axiom 11 is invalid

	Is field?	Reason (if not)
$(\mathbb{Z}, +, \cdot)$	False	Multiplicative inverse doesn't exist
$(\mathbb{Q}, +, \cdot)$	True	
$(\mathbb{Q}^c, +, \cdot)$	False	$\sqrt{2} \cdot \sqrt{2} \notin \mathbb{Q}^c$
Boolean algebra	False	Additive inverse doesn't exist
$(\{0,1\}, + \bmod 2, \cdot \bmod 2)$	True	
$(\{0,1,2\}, + \bmod 3, \cdot \bmod 3)$	True	
$(\{0,1,2,3\}, + \bmod 4, \cdot \bmod 4)$	False	Multiplicative inverse doesn't exist

Completeness Axiom

Let A be a non empty subset of $\mathbb{R}$.

- u is the upper bound of A if: $\forall a \in A; a \leq u$
- A is bounded above if A has an upper bound
- Maximum element of A: $\max A = u$ if $u \in A$ and u is an upper bound of A
- Supremum of A $\sup A$, is the smallest upper bound of A
- Maximum is a supremum. Supremum is not necessarily a maximum.
- l is the lower bound of A if: $\forall a \in A; a \geq l$
- A is bounded below if A has a lower bound
- Minimum element of A: $\min A = l$ if $l \in A$ and l is a lower bound of A
- Infimum of A $\inf A$, is the largest lower bound of A
- Minimum is a infimum. Infimum is not necessarily a minimum.

Theorems

Let A be a non empty subset of $\mathbb{R}$.

- Say u is an upper bound of A. Then $u = \sup A$ **iff**:
  $\forall \epsilon > 0\ \exists a \in A;\ a + \epsilon > u$
- Say l is a lower bound of A. Then $l = \inf A$ **iff**:
  $\forall \epsilon > 0\ \exists a \in A;\ a - \epsilon < l$

Required proofs

- $sup(a, b) = b$
- $inf(a, b) = a$

Completeness axioms of real numbers

- Every non empty subset of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$
- Every non empty subset of $\mathbb{R}$ which is bounded below has a infimum in $\mathbb{R}$

> **Note**
>
> $\mathbb{Q}$ doesn't have the completeness property.

Completeness axioms of integers

- Every non empty subset of $\mathbb{Z}$ which is bounded above has a maximum
- Every non empty subset of $\mathbb{Z}$ which is bounded below has a minimum

Two important theorems

- $\exists a\ \forall \epsilon > 0, a < \epsilon \implies a \leq 0$
- $\forall \epsilon > 0\ \exists a, a < \epsilon \not\Rightarrow a \leq 0$

Order Axioms

- **Trichotomy**: $\forall a, b \in \mathbb{R}$ exactly one of these holds: $a > b$, $a = b$, $a < b$
- **Transitivity**: $\forall a, b, c \in \mathbb{R}; a < b \land b < c \implies a < c$
- **Operation with addition**: $\forall a, b \in \mathbb{R}; a < b \implies a + c < b + c$
- **Operation with mutliplication**: $\forall a, b, c \in \mathbb{R}; a < b \land 0 < c \implies ac < bc$

Definitions

- $a < b \equiv b > a$
- $a \leq b \equiv a < b \lor a = b$
- $a \neq b \equiv a < b \lor a > b$
- $|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$

Triangular inequalities

$$|a| - |b| \leq |a + b| \leq |a| + |b|$$

$$||a| - |b|| \leq |a + b|$$

Required proofs

- $\forall a, b, c \in \mathbb{R}; a < b \wedge c < 0 \implies ac > bc$
- $1 > 0$
- $-|a| \leq a \leq |a|$
- Triangular inequalities

Theorems

- $\exists a \ \forall \epsilon > 0, \ a < \epsilon \implies a \leq 0$
- $\exists a \ \forall \epsilon > 0, \ 0 \leq a < \epsilon \implies a = 0$

> **Caution**
>
> $\forall \epsilon > 0 \ \exists a, \ a < \epsilon \implies a \leq 0$ is **not** valid.

Let A be a non-empty subset of $\mathbb{R}$ which is bounded above and has an upper bound u.

$$u = \sup A \iff \forall \epsilon > 0 \, \exists a \in A, \ a > u - \epsilon$$

Let A be a non-empty subset of $\mathbb{R}$ which is bounded below and has an lower bound m.

$$m = \inf A \iff \forall \epsilon > 0 \, \exists a \in A, \ a < m + \epsilon$$

Relations

Definitions

- Cartesian Product of sets A, B
  $A \times B = \{(a,b) | a \in A, b \in B\}$
- Ordered pair
  $(a,b) = \{\{a\},\{a,b\}\}$

Relation

Let $A, B \neq \emptyset$. A relation $R : A \to B$ is a non-empty subset of $A \times B$.

- $a\,R\,b \equiv (a,b) \in R$
- Domain of R: $dom(R) = A$
- Codomain of R: $codom(R) = B$
- Range of R: $ran(R) = \{y | (x,y) \in R\}$
- $ran(R) \subseteq B$
- Pre-range of R: $preran(R) = \{x \mid (x,y) \in R\}$
- $preran(R) \subseteq A$
- $R(a) = \{b \mid (a,b) \in R\}$

Everywhere defined

R is everywhere defined
$\iff A = dom(R) = preran(R)$
$\iff \forall a \in A,\ \exists b \in B;\ (a,b) \in R$.

Onto

R is onto
$\iff B = codom(R) = ran(R)$
$\iff \forall b \in B\, \exists a \in A\, (a,b) \in R$

Aka. **surjection**.

Inverse

Inverse of R: $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

Types of relation

one-many

$$\iff \exists a \in A, \exists b_1, b_2 \in B\ ((a, b_1), (a, b_2) \in R \wedge b_1 \neq b_2)$$

Not one-many

$$\iff \forall a \in A, \forall b_1, b_2 \in B\ ((a, b_1), (a, b_2) \in R \implies b_1 = b_2)$$

many-one

$$\iff \exists a_1, a_2 \in A, \exists b \in B\ ((a_1, b), (a_2, b) \in R \wedge a_1 \neq a_2)$$

Not many-one

$$\iff \forall a_1, a_2 \in A, \forall b \in B\ ((a_1, b), (a_2, b) \in R \implies a_1 = a_2)$$

many-many

iff R is **one-many** and **many-one**.

one-one

iff R is **not one-many** and **not many-one**. Aka. **injection**.

Bijection

When a relation is **onto** and **one-one**.

Functions

A function $f: A \to B$ is a relation $f: A \to B$ which is everywhere defined and not one-many.

- $dom(f) = A = preran(f)$

Inverse

For a function $f:\ A \to B$ to have its inverse relation $f^{-1}:\ B \to A$ be also a function, we need:

- f is onto
- f is not many-one (in other words, f must be one-one)

The above statement is true for all unrestricted function f that has an inverse f^{-1}:

$$f(f^{-1}(x)) = x = f^{-1}(f(x)) = x$$

Composition

Composition of relations

Let $R : A \to B$ and $S : B \to C$ are 2 relations. Composition can be defined when $\text{ran}(R) = \text{preran}(S)$.

Say $\text{ran}(R) = \text{preran}(S) = D$. Composition of the 2 relations is written as:

$$S \circ R = \{(a, c) \,|\, (a, b) \in R,\ (b, c) \in S,\ b \in D\}$$

Composition of functions

Let $f : A \to B$ and $g : B \to C$ be 2 functions where f is onto.

$$g \circ f = \{(x, z) \,|\, (x, y) \in f,\ (y, z) \in g,\ y \in B\} = g(f(x))$$

Countability

A set A is countable **iff** $\exists f : A \to Z^+$, where f is a one-one function.

Examples

- Countable: Any finite set, $\mathbb{Z}, \mathbb{Q}$
- Uncountable: $\mathbb{R}$, Any open/closed intervals in $\mathbb{R}$.

Transitive property

Say $B \subset A$.

$$A \text{ is countable} \implies B \text{ is countable}$$

$$B \text{ is not countable} \implies A \text{ is not countable}$$

Limits

$\lim_{x \to a} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$

Defining δ in terms of a given ϵ is enough to prove a limit.

One sided limits

$\lim_{x \to a^+} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < x - a < \delta \implies |f(x) - L| < \epsilon)$$

$\lim_{x \to a^-} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (-\delta < x - a < 0 \implies |f(x) - L| < \epsilon)$$

$\lim_{x \to a} f(x) = L^+$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies 0 \leq f(x) - L < \epsilon)$$

$\lim_{x \to a} f(x) = L^-$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies -\epsilon < f(x) - L \leq 0)$$

Limits including infinite

$\lim_{x \to \infty} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \ \exists N > 0 \ \forall x \ (x > N \implies |f(x) - L| < \epsilon)$$

$\lim_{x \to -\infty} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \ \exists N > 0 \ \forall x \ (x < -N \implies |f(x) - L| < \epsilon)$$

$\lim_{x \to a} f(x) = \infty$ **iff**:

$$\forall M > 0 \ \exists \delta > 0 \ \forall x \ (0 < |x - a| < \delta \implies f(x) > M)$$

$\lim_{x \to a} f(x) = -\infty$ **iff**:

$$\forall M > 0 \ \exists \delta > 0 \ \forall x \ (0 < |x - a| < \delta \implies f(x) < -M)$$

Indeterminate forms

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $\infty \cdot 0$
- $\infty - \infty$
- ∞^0
- 0^0
- 1^∞

Continuity

A function f is continuous at a **iff**:

$$\lim_{x \to a} f(x) = f(a)$$

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x\ (|x-a| < \delta \implies |f(x) - f(a)| < \epsilon)$$

One-side continuous

A function f is continuous from right at a **iff**:

$$\lim_{x \to a^+} f(x) = f(a)$$

A function f is continuous from left at a **iff**:

$$\lim_{x \to a^-} f(x) = f(a)$$

Continuous on an open interval

A function f is continuous in (a, b) **iff** f is continuous on every $c \in (a, b)$.

Continuous on a closed interval

A function f is continuous in $[a, b]$ **iff** f is:

- continuous on every $c \in (a, b)$
- right-continuous at a
- left-continuous at b

Uniformly continuous

Suppose a function f is continuous on (a, b). f is uniformly continuous on (a, b) **iff**:

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ s.t. } |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

If a function f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$.

> ⚠ **Todo**
>
> Is this section correct? I am not 100% sure.

Continuity Theorems

Extreme Value Theorem

If f is continuous on $[a, b]$, f has a maximum and a minimum in $[a, b]$.

> ⓘ **Proof Hint**
>
> Proof is quite hard.

Intermediate Value Theorem

Let f is continuous on $[a, b]$. If $\exists u$ such that $f(a) > u > f(b)$ or $f(a) < u < f(b)$: $\exists c \in (a, b)$ such that $f(c) = u$.

> ⓘ **Proof Hint**
>
> Proof the case when $u = 0$. Otherwise define a new function $g(x)$ such that middle part of the above inequality has a 0 in the place of u.

Sandwich (or Squeeze) Theorem

Let:

- For some $\delta > 0$: $\forall x(0 < |x - a| < \delta \implies f(x) \leq g(x) \leq h(x))$
- $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \in \mathbb{R}$

Then $\lim_{x \to a} g(x) = L$.

> ⓘ **Note**
>
> Works for any kind of x limits.

"No sudden changes"

Positive

Let f be continuous on a and $f(a) > 0$

$$\implies \exists \delta > 0; \forall x \left(|x - a| < \delta \implies f(x) > 0\right)$$

> ⓘ **Proof Hint**
>
> To proof this, take $\epsilon = \frac{f(a)}{2}$.

Negative

Let f be continuous on a and $f(a) < 0$

$\implies \exists\delta > 0; \forall x\,(|x-a| < \delta \implies f(x) < 0)$

> ⓘ **Proof Hint**
>
> To proof this, take $\epsilon = -\frac{f(a)}{2}$.

Differentiability

A function f is differentiable at a **iff**:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'(a)$$

$f'(a)$ is called the derivative of f at a.

One-side differentiable

Left differentiable

A function f is left-differentiable at a **iff**:

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'_-(a)$$

Right differentiable

A function f is right-differentiable at a **iff**:

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'_+(a)$$

Differentiability implies continuity

$$f \text{ is differentiable at } a \implies f \text{ is continuous at } a$$

> ⓘ **Proof Hint**
>
> Use $\delta = min(\delta_1, \frac{\epsilon}{1+|f'(a)|})$.

> ⓘ **Note**
>
> Suppose f is differentiable at a. Define g:
>
> $$g(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a}, & x \neq a \\ f'(a), & x = a \end{cases}$$
>
> g is continuous at a.

Extreme Values

Suppose $f : [a, b] \to \mathbb{R}$, and $F = f([a, b]) = \left\{ f(x) \mid x \in [a, b] \right\}$. Minimum and maximum values of f are called the extreme values.

Maximum

Maximum of the function f is $f(c)$ where $c \in [a, b]$ **iff**:

$$\forall x \in [a, b], \; f(c) \geq f(x)$$

aka. **Global Maximum**. Maximum doesn't exist always.

Local Maximum

A Local maximum of the function f is $f(c)$ where $c \in [a, b]$ **iff**:

$$\exists\delta\ \ \forall x\,(0<|x-c|<\delta \implies f(c)\geq f(x))$$

Global maximum is obviously a local maximum.

The above statement can be simplified when $c=a$ or $c=b$.

When $c=a$:

$$\exists\delta\ \ \forall x\,(0<x-c<\delta \implies f(c)\geq f(x))$$

When $c=b$:

$$\exists\delta\ \ \forall x\,(-\delta<x-c<0 \implies f(c)\geq f(x))$$

Minimum

Minimum of the function f is $f(c)$ where $c\in[a,b]$ **iff**:

$$\forall x\in[a,b],\ f(c)\leq f(x)$$

aka. **Global Minimum**. Minimum doesn't exist always.

Local Minimum

$$\exists\delta\ \ \forall x\,(0<|x-c|<\delta \implies f(c)\leq f(x))$$

Global minimum is obviously a local maximum.

The above statement can be simplified when $c=a$ or $c=b$.

When $c=a$:

$$\exists\delta\ \ \forall x\,(0<x-c<\delta \implies f(c)\leq f(x))$$

When $c=b$:

$$\exists \delta \ \forall x \, (-\delta < x - c < 0 \implies f(c) \leq f(x))$$

Special cases

f is continuous

Then by Extreme Value Theorem, we know f has a minimum and maximum in $[a, b]$.

f is differentiable

- If $f(a)$ is a local maximum: $f'_+(a) \leq 0$
- If $f(b)$ is a local maximum: $f'_-(b) \geq 0$
- $c \in (a, b)$ and If $f(c)$ is a local maximum: $f'(c) = 0$
- If $f(a)$ is a local minimum: $f'_+(a) \geq 0$
- If $f(b)$ is a local minimum: $f'_-(b) \leq 0$
- $c \in (a, b)$ and If $f(c)$ is a local minimum: $f'(c) = 0$

Critical point

$c \in [a, b]$ is called a critical point **iff**:

$$f'(c) = 0 \quad \lor \quad f'(c) \text{ is undefined}$$

Other Theorems

Rolle's Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). And $f(a) = f(b)$. Then:

$$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$$

> ⓘ **Proof Hint**
>
> By Extreme Value Theorem, maximum and minimum exists for f.
>
> Consider 2 cases:
>
> 1. Both minimum and maximum exist at a and b.

> 2. One of minimum or maximum occurs in (a, b) .

Mean Value Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then:

$$\exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

> **Proof Hint**
>
> - Define $g(x) = f(x) - \left(\frac{f(a)-f(b)}{a-b}\right)x$
> - $g(a)$ will be equal to $g(b)$
> - Use Rolle's Theorem for g

Cauchy's Mean Value Theorem

Let f and g be continuous on $[a, b]$ and differentiable on (a, b), and $\forall x \in (a, b)\ g'(x) \neq 0$ Then:

$$\exists c \in (a, b) \text{ s.t. } \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

> **Proof Hint**
>
> - Define $h(x) = f(x) - \left(\frac{f(a)-f(b)}{g(a)-g(b)}\right)g(x)$
> - $h(a)$ will be equal to $h(b)$
> - Use Rolle's Theorem for h

This is a more generalized version of the mean value theorem. Mean value theorem is the case when $g(x) = x$.

> **Note**

> L'Hopital's rule can be proven using Cauchy's Mean Value Theorem.

Generalized MVT for Riemann Integrals

Let f, g be continuous on $[a, b]$ ($\implies$ f, g are integrable), and g does not change sign on (a, b). Then $\exists \zeta \in (a, b)$ such that:

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\zeta)\int_a^b g(x)\mathrm{d}x$$

> **Proof Hint**
>
> - Use Extreme value theorem for f
> - Multiply by $g(x)$. Then integrate. Then divide by $\int_a^b g(x)$.
> - Use intermediate value theorem to find $f(\zeta)$

L'Hopital's Rule

> **Note**
>
> Be careful with the pronunciation.
>
> - It's not "Hospital's Rule", there are no "s"
> - It's not "Hopital's Rule" either, there is a "L'".

L'Hopital's Rule can be used when all of these conditions are met. (here δ is some positive number).

1. f and g are 2 functions defined at a
2. $f(a) = g(a) = 0$
   Also valid when either of these conditions is satisfied
   - $\lim f(x) = \lim g(x) = 0$
   - $\lim f(x) = \lim g(x) = \infty$
3. f, g are continuous on $x \in [a, a+\delta]$
4. f, g are differentiable on $x \in (a, a+\delta)$

5. $g'(x) \neq 0$ on $x \in (a, a+\delta)$
6. $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$

Then: $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$

> **Note**
>
> L'Hopital's Rule is valid for all types of "x limits".

Higher order derivatives

Suppose f is a function defined on (a, b). f is n times differentiable or n-th differentiable **iff**:

$$\lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x - a} = L \in \mathbb{R} = f^{(n)}(a)$$

Here $f^{(n)}$ denotes n-th derivative of f. And $f^{(0)}$ means the function itself.

$f^{(n)}(a)$ is the n-th derivative of f at a.

> **Note**
>
> $$f^{(n)} \text{ is differentiable at } a \implies f^{(n-1)} \text{ is continuous at } a$$

Taylor's Theorem

Let f is $n+1$ differentiable on (a, b). Let $c, x \in (a, b)$. Then $\exists \zeta \text{ s.t.}$:

$$f(x) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-c)^{n+1}$$

[Mean value theorem](#) can be derived from taylor's theorem when $n = 0$.

> ⓘ **Proof Hint**
>
> - Define $F(t) = f(t) + \sum\_k = 1^n \frac{f^{(k)}(t)}{k!}(x-t)^k$
> - Define $G(t) = (x-t)^{n+1}$
> - Consider the interval $[c, x]$
> - Use Cauchy's mean value theorem for F, G after making sure the conditions are met.

The above equation can be written like:

$$f(x) = T_n(x, c) + R_n(x, c)$$

Taylor Polynomial

This part of the above equation is called the Taylor polynomial. Denoted by $T_n(x, c)$.

$$T_n(x, c) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Remainder

Denoted by $R_n(x, c)$.

$$R_n(x, c) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-c)^{n+1}$$

Integral form of the remainder

$$R_n(x, c) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n \mathrm{d}t$$

> ⓘ **Proof Hint**

- Method 1: Use integration by parts and mathematical induction.
- Method 2: Use Generalized MVT for Riemann Integrals where:
    - $F = f^{(n+1)}$
    - $G = (x - t)^n$

Note

When $n = 1$:

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(\zeta)}{2!}(x - c)^2$$

$$f(x) - \text{Tangent line} = \frac{f''(\zeta)}{2!}(x - c)^2$$

From this: $f''(c) > 0 \implies$ a local minimum is at c. Converse is **not** true.