Summary | Riemann Integration

Introduction

Interval

Let I = [a, b]. Length of the interval |I| = b - a.

Disjoint interval

When 2 intervals don't share any common numbers.

Almost disjoint interval

When 2 intervals are disjoint or intersect only at a common endpoint.

Riemann Integral

Let $f - [a, b] \rightarrow \mathbb{R}$ is a bounded (not necessarily continuous) function on a closed, bounded (compact) interval.

Riemann integral of $oldsymbol{f}$ is: $\int_a^b oldsymbol{f}$

Definite integral

When a, b are constants.

Indefinite integral

When a is a constant but b is replaced with x.

Partition

Let I be a non-empty, compact interval (closed and bounded). A partition of I is a finite collection $\{I_1, I_2, \ldots, I_n\}$ of almost disjoint, non-empty, compact sub-intervals whose union is I.

A partition is determined by the endpoints of all sub-intervals:

 $a = x_0 < x_1 < \cdots < x_n = b$.

A partition can be denoted by:

- its intervals $P = \{I_1, I_2, \dots, I_n\}$
- the endpoints of its intervals $P=\{x_0,x_1,\ldots,x_n\}$

Riemann Sum

Let

- $f:[a,b] o \mathbb{R}$ is a bounded function on the compact interval I=[a,b] with $M=\sup_I f$ and $m=\inf_I f$.

•
$$P = \{I_1, I_2, \ldots, I_n\}$$

$$\cdot \hspace{0.2cm} M_k = \sup_{I_k} f = \sup \left\{ f(x) : x \in [x_{k-1}, x_k] \right\}$$

 $\cdot \ \ m_k = \inf_{I_k} f = \inf \left\{ f(x) : x \in [x_{k-1}, x_k]
ight\}$

Upper riemann sum

$$U(f;P) = \sum_{k=1}^n M_k |I_k|$$

Lower riemann sum

$$L(f;P) = \sum_{k=1}^n m_k |I_k|$$

 $m_k < M_k \implies L(f;P) \le U(f;P)$

When P_1, P_2 are any 2 partitions of I: $L(f;P_1) \leq U(f;P_2)$

Refinements

Q is called a refinement of $P\iff$ if P and Q are partitions of [a,b] and $P\subseteq Q.$ When Q is a refinement of P:

$$L(f;P) \leq L(f;Q) \leq U(f;Q) \leq U(f;P)$$

(i) Note

If P_1 and P_2 are partitions of [a,b], then $Q=P_1\cup P_2$ is a refinement of both P_1 and P_2 . In that case:

$$L(f;P_1) \leq L(f;Q) \leq U(f;Q) \leq U(f;P_2)$$

Upper & Lower integral

Let \mathbb{P} be the collection of all possible partitions of the interval [a, b].

Upper Integral

$$U(f) = \inf \left\{ U(f;P); P \in \mathbb{P}
ight\} = \overline{\int_a^b f}$$

Lower Integral

$$L(f) = \sup \left\{ L(f;P); P \in \mathbb{P}
ight\} = {\displaystyle \int_{a}^{b} f}$$

For a bounded function f, always $L(f) \leq U(f)$

Riemann Integrable

A bounded function $f:[a,b] o \mathbb{R}$ is Riemann integrable on [a,b] iff U(f) = L(f). In that case, the Riemann integral of f on [a,b] is denoted by $\int_a^b f(x) \, \mathrm{d} x$.

Reimann Integrable or not

Function	Yes or No?	Proof hint
Unbounded	No	By definition
Constant	Yes	$orall P\left(ext{any partition} ight) \; L(f;P) = U(f;P)$
Monotonically increasing/decreasing	Yes	Take a partition such that $\Delta x < \delta = rac{\epsilon}{f(b)-f(a)}$
Continuous	Yes	Take a partition such that $\Delta x < \delta = rac{\epsilon}{2(b-a)}$

(i) Note

If the set of points of discontinuity of a bounded function $f:[a,b] \to \mathbb{R}$ is finite, then f is Riemann integrable on [a,b].

(i) Note

If the set of points of discontinuity of a bounded function $f:[a,b] \to \mathbb{R}$ is finite number of limit points, then f is integrable on [a,b].

A function may have infinitely many discontinuous points, but if the set of all discontinuous points have finite number of limit points, then f is integrable on [a, b].

Cauchy Criterion

Theorem

A bounded function $f:[a,b] \to R$ is Riemann integrable **iff** for every $\epsilon > 0$ there exists a partition P_{ϵ} of [a,b], which may depend on ϵ , such that:

. . . .

$U(f,P\epsilon)-L(f,P\epsilon)\leq\epsilon$

(i) Proof Hint

- To prove \implies : consider $L(f)-rac{\epsilon}{2}$ and $U(f)+rac{\epsilon}{2}$
- To prove \iff : consider $L(f;P) < L(f) \land U(f) < U(f;P)$

(i) Note

 $f:[a,b]
ightarrow\mathbb{R}$ is integrable on [a,b] when:

- The set of points of discontinuity of a bounded function $\,f\,$ is finite.
- The set of points of discontinuity of a bounded function f is finite number of limit points. (may have infinite number of discontinuities) :::

Theorems on Integrability

Theorem 1

Suppose $f: [a, b] \to \mathbb{R}$ is bounded, and integrable on [c, b] for all $c \in (a, b)$. Then f is integrable on [a, b]. Also valid for the other end.

(i) Proof Hint

- Isolate a partition on the required end.
- Choose x_1 or x_{n-1} such that $\Delta x < rac{\epsilon}{4M}$ where M is an upper or lower bound.

Theorem 2

Suppose $f: [a, b] \to \mathbb{R}$ is bounded, and continuous on [c, b] for all $c \in (a, b)$. Then f is integrable on [a, b]. Also valid for the other end.

A TODO: Proof Hint

Properties of Integrals

Notation

If a < b and f is integrable on [a, b], then:

$$\int_a^b f = -\int_b^a f$$

Properties

Suppose f and g are integrable on [a, b].

Addition

f + g will be integrable on [a, b].

$$\int_a^b (f\pm g) = \int_a^b f\pm \int_a^b g$$

(i) Proof Hint

- Prove f+g is integrable using: • $sup(f+g) \leq \sup(f) + \sup(g)$)
 - $inf(f+g) \geq \inf(f) + \inf(g)$)
- Start with $\, U(f+g)\,$ and show $\, U(f+g) \leq U(f) + U(g)\,$
- Start with L(f+g) and show $L(f+g) \geq L(f) + L(g)$

Constant multiplication

Suppose $k \in \mathbb{R}$. kf will be integrable [a, b].

$$\int_a^b kf = k\int_a^b f$$

(i) Proof Hint

- Prove for $k\geq 0$. Use $U-L<rac{\epsilon}{k}$
- Prove for $\,k=-1\,$
- Using the above results, proof for $\,k < 0\,$ is apparent

Bounds

If $m \leq f(x) \leq M$ on [a,b]:

$$m \leq \int_a^b f \leq M$$

If $f(x) \leq g(x)$ on [a,b]:

$$\int_a^b f \leq \int_a^b g$$

Modulus

|f| will be integrable on [a, b].

$$\left|\int_{a}^{b}f
ight|=\int_{a}^{b}\left|f
ight|$$

(i) Proof Hint

Start with $-|f| \leq f \leq |f|$. And integrate both sides.

Multiple

fg will be integrable on [a,b].

(i) Proof Hint

- Suppose $oldsymbol{f}$ is bounded by $oldsymbol{k}$
- Prove f^2 is integrable (Use $rac{\epsilon}{2k}$)
- fg is integrable because:

$$fg=rac{1}{2}ig[(f+g)^2-f^2-g^2ig]$$

Max, Min

 $\max(f,g)$ and $\min(f,g)$ are integrable.

Where \max and \min functions are defined as:

$$\max(f,g) = rac{1}{2}(|f-g|+f+g)$$
 $\min(f,g) = rac{1}{2}(-|f-g|+f+g)$

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